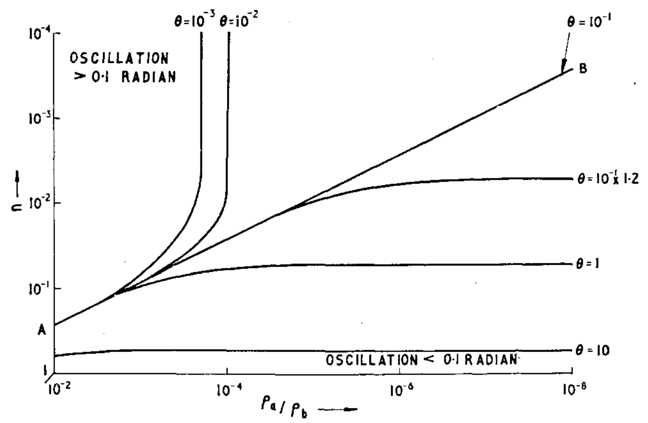

Fig. 2 Relation between spin, gun jump, and air density for the yawing amplitude to be less than 0.1 rad; $c = 10$, $\tau = 4000$.

the line AB and then breaks away to asymptote at a critical density. With such low values of θ and for densities lower than the critical no spin is required to restrict the oscillation of 0.1 rad. The critical density ratio for a given gun jump θ is given by

$$(\rho_a/\rho_b) = (\pi/48)[c^3\gamma(\theta)^2/\tau^2] \tag{23}$$

If the gun jump parameter θ is greater than 10^{-1} then in order to restrict the yawing to less than 0.1 rad. n must be increased until the spin density point lies in the region of stable oscillations and below the line for the gun jump parameter. At sufficiently low densities this line is independent of density. This density is given approximately by

$$(\rho_a/\rho_b) = (\pi/48)[c^3\delta(\theta)^2/\tau^2] \tag{24}$$

and the minimum spin is then

$$n = (2/3\pi)(c^2\delta/\tau) \tag{25}$$

Conclusion

In a hyperballistic range where the projectile velocity is high, the time of flight short and the range density often quite low the equations of motion of a spinning projectile may take a very simple form whose solution is given in Eq. (4). Examining this solution shows that the classical stability criterion is not sufficient to ensure that the yawing oscillations are small. Revised criteria which restrict the yawing oscillation to ϵ radian are given in Eqs. (10) and (14).

Detailed calculations are presented in Figs. 1 and 2 for a range length $\tau(= R/d)$ of 4000 and projectile (length/diam) ratios of 3 and 10. It has been shown in these cases that yawing can theoretically be restricted to less than 0.1 radian provided the following conditions hold.

1) If the gun jump parameter $\theta(= R\dot{\alpha}/V)$ is less than 0.1 then the minimum spin at a given density must correspond to a point on the curves of appropriate θ above the line AB. There exists a critical density, given by Eq. (23), below which it is unnecessary to spin the projectile at all for that amount of gun jump.

2) If the gun jump parameter θ is greater than 0.1 then the minimum spin at a given density must correspond to a point on the curve of appropriate θ below the line AB. At densities below the value given in Eq. (24) the minimum spin is constant at the value given in Eq. (25).

References

[1] Arnold, R. N. and Maunder, L., *Gyrodynamics*, Academic Press, New York, 1961.

[2] Fowler, R. H., Gallop, E. G., Lock, C. N. H., and Richmond, H. W., *Philosophical Transactions of the Royal Society, Series A*, Vol. 221, 295, 1920.

Twisting of a Composite Spherically Aeolotropic Sphere with an Isotropic Core

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Introduction

In this paper the problem of twisting of a sphere of a spherically aeolotropic material is considered when there is an isotropic core. In the shell the planes of symmetry at each point are taken to be perpendicular to the directions of spherical polar coordinates (r,θ,ϕ). The shell is bounded by surfaces $r = a$ and $r = b$. The composite body is twisted by tangential forces on the surface $r = b$, the traction on the hemispherical surface $0 \leqslant \theta \leqslant \pi/2$ being equivalent to a couple about the polar axis from which θ is measured. Stresses are found in the aeolotropic shell as also in the isotropic core. Particular cases are deduced 1) when there is a cavity inside and 2) when the inclusion is rigid. The first case was given by Chakravorty[1] without any numerical computation. Extensive numerical results are given showing the concentration of stress on the boundary $r = a$ of the inclusion when the outer material is topaz (spherically aeolotropic) and the inner material is steel (isotropic). It is seen that the stress concentration is maximum at the equatorial region. Effects of cavity or rigid inclusion, on stresses, are shown in tabular form.

Problem and Its Solution

We use spherical polar coordinates r,θ,ϕ, the centre of the shell being taken as the pole. The strain energy function of such an aeolotropic material is given by

$$2W = c_{11}e_{rr}^2 + c_{22}e_{\theta\theta}^2 + c_{33}e_{\phi\phi}^2 + 2c_{23}e_{\theta\theta}e_{\phi\phi} + 2c_{13}e_{\phi\phi}e_{rr} + 2c_{12}e_{rr}e_{\theta\theta} + c_{44}e_{\theta\phi}^2 + c_{55}e_{\phi r}^2 + c_{66}e_{r\theta}^2 \tag{1}$$

where c_{ij}'s are elastic coefficients.

For twisting of the body by couples about the polar axis from which θ is measured, we assume the displacement components as

$$u = v = 0, w = R \sin 2\theta \tag{2}$$

where R is a function of r only. For the outer shell which is spherically aeolotropic the stresses are

$$\begin{aligned} \sigma_r &= \sigma_\theta = \sigma_\phi = \tau_{r\theta} = 0 \\ \tau_{\theta\phi} &= -2c_{44}(R/r)\sin^2\theta \\ \tau_{\phi r} &= c_{55}(dR/dr - R/r)\sin 2\theta \end{aligned} \tag{3}$$

Table 1 Values of $-\pi a^3\tau_{\theta\phi i}/M$ for certain values of θ and δ in the composite case

	δ				
θ	0.2	0.4	0.6	0.8	1.0
15°	0.0014	0.0029	0.0043	0.0058	0.0072
30°	0.0054	0.0108	0.0161	0.0215	0.0269
45°	0.0108	0.0215	0.0323	0.0430	0.0538
60°	0.0161	0.0323	0.0484	0.0646	0.0807
75°	0.0201	0.0402	0.0602	0.0803	0.1004
90°	0.0213	0.0426	0.0640	0.0853	0.1067

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Table 2 Values of $-\pi a^3\tau_{\theta\phi 0}/M$ for certain values of θ and δ in the composite case

θ	δ					
	1.0	1.2	1.4	1.6	1.8	2.0
15°	0.0095	0.0108	0.0121	0.0135	0.0149	0.0163
30°	0.0354	0.0402	0.0453	0.0505	0.0557	0.0608
45°	0.0709	0.0804	0.0906	0.1010	0.1113	0.1216
60°	0.1063	0.1206	0.1360	0.1515	0.1670	0.1823
75°	0.1322	0.1500	0.1690	0.1883	0.2076	0.2267
90°	0.1418	0.1608	0.1812	0.2020	0.2226	0.2431

The corresponding nonvanishing stresses in the isotropic core are

$$\begin{aligned}\tau_{\theta\phi} &= -2\mu(R/r)\sin^2\theta \\ \tau_{\phi r} &= \mu(dR/dr - R/r)\sin 2\theta\end{aligned} \tag{4}$$

μ being the shear modulus of the isotropic inclusion. Two of the equations of equilibrium[2] are identically satisfied by (3) while the third reduces to

$$d^2R/dr^2 + (2/r)dR/dr - 2(c_{55} + 2c_{44})/c_{55}(R/r^2) = 0$$

whence

$$R = Ar^{n-1/2} + Br^{-n-1/2} \tag{5}$$

where

$$n = (\tfrac{1}{2})[1 + 8(c_{55} + 2c_{44})/c_{55}]^{1/2}$$

In the associated isotropic core

$$R = Cr^2 \tag{6}$$

Henceforth we shall use the subscript o and i to denote the corresponding displacements and stresses in the aeolotropic outer shell and the isotropic inner core. Thus from (5) and (6) by (3) and (4) we have

$$\begin{aligned}w_0 &= [(Ar^{n-1/2} + Br^{-n-1/2})]\sin 2\theta \\ w_i &= Cr^2 \sin 2\theta\end{aligned} \tag{7}$$

$$\begin{aligned}\tau_{\theta\phi 0} &= -2c_{44}[(Ar^{n-3/2} + Br^{-n-3/2})]\sin^2\theta \\ \tau_{\theta\phi i} &= -2\mu Cr \sin^2\theta\end{aligned} \tag{8}$$

$$\begin{aligned}\tau_{\phi r0} &= c_{55}[A(n - \tfrac{3}{2})r^{n-3/2} - B(n + \tfrac{3}{2})r^{-n-3/2}]\sin 2\theta \\ \tau_{\phi ri} &= \mu Cr \sin 2\theta\end{aligned} \tag{9}$$

Boundary conditions:

1) $w_i = w_0$ on $r = a$

2) $\tau_{\phi ri} = \tau_{\phi r0}$ on $r = a$ (10)

3) $M = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} [\tau_{\phi r0}]_{r=b}\cdot b^2 \sin\theta d\theta d\phi\cdot b\sin\theta$

The third boundary condition implies that the tractions on the hemispherical surface $0 \leq \theta \leq \pi/2$ on $r = b$ are statically equivalent to a couple about the polar axis of moment M. The tractions on the other hemisphere $r = b, \pi/2 \leq \theta \leq \pi$ are statically equivalent to an equal couple about the same axis but in opposite direction.

Table 3 Values of $(\tau_{\theta\phi i} - \tau_{\theta\phi 0})\pi a^3/M$ for certain values of θ when $\delta = 1$

15°	30°	45°	60°	75°	90°
0.0023	0.0085	0.0171	0.0256	0.0318	0.0351

Table 4 Numerical values of $\pi a^3\tau_{\phi r}/M = P$ at $\delta = 45°$ for certain values of θ

P[a]	δ					
	1.0	1.2	1.4	1.6	1.8	2.0
P_i	0.0538	0.0737	0.0888	0.1017	0.1136	0.1250
P_c	0.0000	0.0485	0.0763	0.0958	0.1114	0.1250
P_r	0.3276	0.2021	0.1522	0.1320	0.1252	0.1250

[a] The subscripts i, c, and r are used with P to denote its values when the inclusion is isotropic, cavity rigid, respectively.

When subjected to the boundary conditions (10), we get

$$\begin{aligned}A &= K(n + 3/2 + \mu/c_{55})a^{-n+5/2} \\ B &= K(n - 3/2 - \mu/c_{55})a^{n+5/2} \\ C &= 2nK\end{aligned} \tag{11}$$

while M being given, K is determined from

$$2M = \pi c_{55}Ka^{5/2}b^{3/2}[(2n - 3)(n + 3/2 + \mu/c_{55})(b/a)^n - (2n + 3)(n - 3/2 - \mu/c_{55})(a/b)^n] \tag{12}$$

A, B, C being determined from (11) and (12), the stresses and displacements are given by (7, 8, and 9).

Case 1

Instead of an isotropic inclusion, if there be a spherical cavity, we get the stresses by making $\mu \to 0$ as

$$\begin{aligned}\tau_{\phi r0} &= K/4\cdot C_{55}a(4n^2 - 9)[(r/a)^{n-3/2} - (a/r)^{n+3/2}]\sin 2\theta \\ \tau_{\theta\phi 0} &= -Kc_{44}a[(2n + 3)(r/a)^{n-3/2} + (2n - 3)(a/r)^{n+3/2}]\sin^2\theta\end{aligned} \tag{13}$$

where K is given by

$$4M = \pi c_{55}Ka^4(4n^2 - 9)[(b/a)^{n+3/2} - (a/b)^{n-3/2}] \tag{14}$$

Case 2

Instead of an isotropic inclusion, if the inclusion be rigid, we get the stresses by making $\mu \to \infty$ as

$$\tau_{\phi r0} = (s/2)[(2n - 3)(r/a)^{n-3/2} + (2n + 3)(a/r)^{n+3/2}]\sin 2\theta$$

$$\tau_{\theta\phi 0} = -2sc_{44}/c_{55}\cdot[(r/a)^{n-3/2} - (a/r)^{n+3/2}]\sin^2\theta \tag{15}$$

$$2M = \pi sa^{3/2}b^{3/2}[(2n - 3)(b/a)^n + (2n + 3)(a/b)^n] \tag{16}$$

Numerical Discussion

We assume $b/a = 2$ and $r/a = \delta$. The material of the outer shell is topaz which is a rhombic crystal for which $c_{44} = 1.10 \times 10^9$ g wt per cm², $c_{55} = 1.35 \times 10^9$ g wt per cm² and the material of the inner core is steel for which $\mu = 8.19 \times 10^{11}$ dynes per cm², (1 g wt = 981 dynes).

Comparing Tables 2 and 5 it is seen that $\tau_{\theta\phi}$ in the shell is greater when there is a cavity inside than when there is an isotropic core. But the effect is just reversed for $\tau_{\phi r}$ as is observed from Table 4.

Table 5 Values of $-\pi a^3\tau_{\theta\phi}/M$ for certain values of θ, δ when there is a cavity inside

θ	δ					
	1.0	1.2	1.4	1.6	1.8	2.0
15°	0.0114	0.0119	0.0129	0.0142	0.0155	0.0169
30°	0.0424	0.0444	0.0484	0.0530	0.0580	0.0631
45°	0.0849	0.0888	0.0967	0.1061	0.1160	0.1262
60°	0.1273	0.1332	0.1450	0.1591	0.1741	0.1893
75°	0.1584	0.1657	0.1805	0.1980	0.2165	0.2355
90°	0.1698	0.1776	0.1934	0.2122	0.2321	0.2524

References

[1] Chakravorty, J. G., "On the Twisting of a Spherical Shell of a Spherically Aeolotropic Material in the Broader Sense," *Journal of the Association of Applied Physicists*, Vol. 3, No. 1, April 1956, pp. 14–16.

[2] Love, A. E. H., "The Mathematical Theory of Elasticity," 4th ed., Dover Publications, New York, 1944.

Self-Preservation in Fully Expanded Turbulent Coflowing Jets

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Introduction

THE problem of the flow in a round jet, when the surrounding air is at rest, has been fully investigated by many authors.[1,2] The present work analyzes more recent data and finds that correlations suggested by Boynton are valid for a wide range of experiments. Furthermore, the correlations are extended to coflowing jets, such as those existing in a low altitude exhaust plume of an ascending missile. The general situation of interest considers axisymmetric turbulent mixing between an inner stream of fluid injected parallel to a moving outer stream.

Jet Mixing with Stagnant Air

The classical analytical treatment[1,2] of jet mixing considers three main regions as shown in Fig. 1. These consist of 1) potential core, 2) the transitional or developing region, and 3) the similarity region, where suitably scaled profiles are self-preserving with axial distance.

It is well established that radial velocity profiles obtained at different axial locations in the fully developed region of subsonic, constant density, axially symmetric, and turbulent free jets issuing into a stagnant medium can be normalized to congruent curves by plotting them in the proper reduced coordinates.[3]

$$(U_{(r)}/U_C) \propto f_1(r/x)^2$$

Where $U_{(r)}$ is the velocity measured at a given radial distance (r) from the centerline, U_C is the centerline velocity and x is the axial distance along the centerline. All points obtained downstream a certain axial distance from the origin,

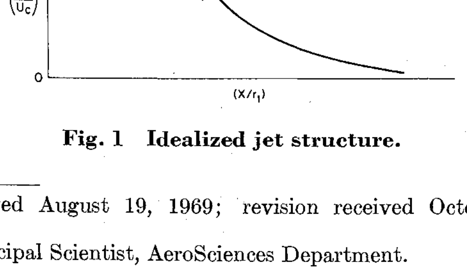

Fig. 1 Idealized jet structure.

i.e., in the similarity region, fall on a universal curve. Such self-preserving flow cannot exist in the "real" coordinate system where significant density variations occur in the flowfield. Furthermore, it has been observed experimentally that supersonic jets tend to spread less than low speed jets. Low speed variable density flows are self-preserving in the Howarth-Dorodnitsyn coordinates, where a transformation is performed on the radial distance.

$$R = \left[2\int_0^r \left(\frac{\rho}{\rho_\infty}\right) r'dr'\right]^{1/2}$$

Where ρ is the density at a point and ρ_∞ is the ambient density. This transformation[4] converts the momentum and continuity equations to their incompressible form. Furthermore, it is necessary to postulate that the supersonic jets can be scaled using a spreading constant, which is a function of the Mach number. It has been found[5] that their scaling constant σ represents the square root of the ratio of the initial total enthalpies to the initial static enthalpies.

$$[\sigma = h_i^0/h_i]^{1/2} = [1 + 1/2(\gamma - 1)M_i^2]^{1/2}$$

A similarity analysis applied[5] to the transformed momentum equation leads to the conclusion that in the fully developed region

$$(U_C/U_i) = (\rho_\infty/\rho_i)^{-1/2} I^{-1/2} \sigma r_i/x$$

where I is the momentum flux defined by

$$I = 2\int_0^\infty f_2{}^2(\eta)\eta d\eta$$

$\eta = R\sigma/x$ and $f_2(\eta)$ is the universal function represented by the low speed data, and subscript i, means the initial conditions (jet nozzle exit). Furthermore, it is assumed that the momentum flux is the same as for the low speed incompressible flows[1] i.e. $I = \frac{1}{4}$.

Data from several sources have been studied by Boynton.[5] This present study applies these findings to additional high and low speed, hot and cold experimental flows.[6] Correlation has been found to be excellent and is given in Fig. 2.

Effect of the Moving External Stream

It has been previously shown[7] for free jets mixing with a moving ambient environment that the excess dimensionless velocity profile (that velocity over the unperturbed external flow) is the same universal function as for the case of a stagnant environment that is in the developed region

$$(U - U_\infty)/(U_C - U_\infty) = \exp[-0.692(r/r_2)^2]$$

where r_2 = position where $U = \frac{1}{2}U_C$.

Theoretical studies were reported in Ref. 8 but their result is inconvenient to apply. Experimental studies of the decay of a jet exhausting into a moving coaxial stream were performed in Ref. 7 by using streams of unequal compositions. It was noted that the presence of a moving external stream

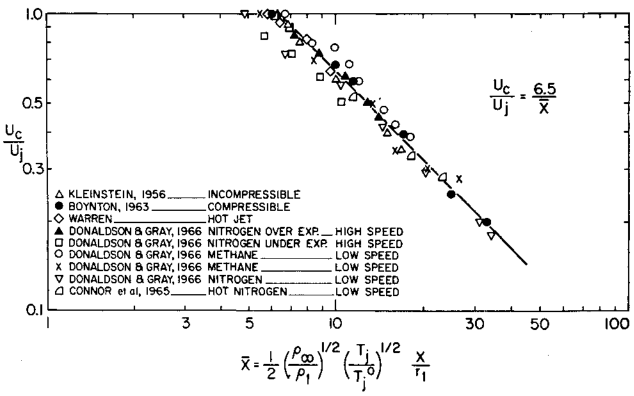

Fig. 2 Decay of centerline velocity.

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